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AUTHOR	Coburn, John W.
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ABSTRACT

This paper describes "The Migration Method," a new technique for graphing a quadratic function (one of the major foundations of undergraduate mathematics) and finding its zeroes. The method promotes a higher degree of conceptualization, reinforces the concept of symmetry, encourages mental visualization, and strengthens the ability to graph a given function by translating a parent or base function. (YDS)

After Five Hundred Years ...
A Significant New Look at Quadratic Functions

Introducing ***The Migration Method*** ...

A new approach to graphing a quadratic function

and determining its zeroes, developed by

John W. Coburn
of
St. Louis Community College
at Florisssant Valley

3400 Pershall Road
St. Louis, Missouri 63135
(314) 595-4343 (office)
(314) 595-2199 (facsimile)
jcoburn@stlcc.edu

A Significant New Look at Quadratic Functions

Introducing ***The Migration Method*** ...

Introduction

Two years ago, a contract was signed with Harcourt College Publishers for the development of a new series of textbooks in developmental mathematics (PreAlgebra through Intermediate Algebra). Writing these textbooks has been a daunting task, yet one that has provided many surprising turns and diverse rewards. I wish to share one of the more substantial "surprises" in this dialogue. Long before pen was put to paper, I made a conscious decision to present and develop the graph of a function *prior* to solving any of the related inequalities. Having always been bothered by the "blind" interval tests seen in many texts, I set about laying groundwork that might enable students to "see" solutions mentally, understand them more completely and solve inequalities with far fewer tests. Linear functions presented no challenge, as a student need only find the x intercept and consider the slope of the line. Likewise, a "visual" solution to absolute value inequalities can easily be developed from characteristics of the graph. However, this presented the challenge of discussing absolute value inequalities *before the topic of translations had been addressed.* While immersed in the consideration of this function and the characteristics of its' graph, I was struck once again by its similarity to the graph of a quadratic function. Noting particularly that the rate of change between a vertical shift of $f(x) = |x|$ and the zeroes of this function was constant, I reasoned that perhaps a similar relationship existed between a vertical shift of $f(x) = x^2$ and its zeroes. The resulting analysis has opened doors and yielded ideas that have apparently not been opened or explored previously. This paper describes a new technique for graphing a quadratic function and finding its zeroes, which I have dubbed ***The Migration Method*** for reasons that will soon be apparent.

For the quadratic function $f(x) = ax^2 + bx + c$,

the following statements are universally accepted.

① The graph of **f** is a parabola

② The parabola is concave up if $a > 0$; concave down if $a < 0$

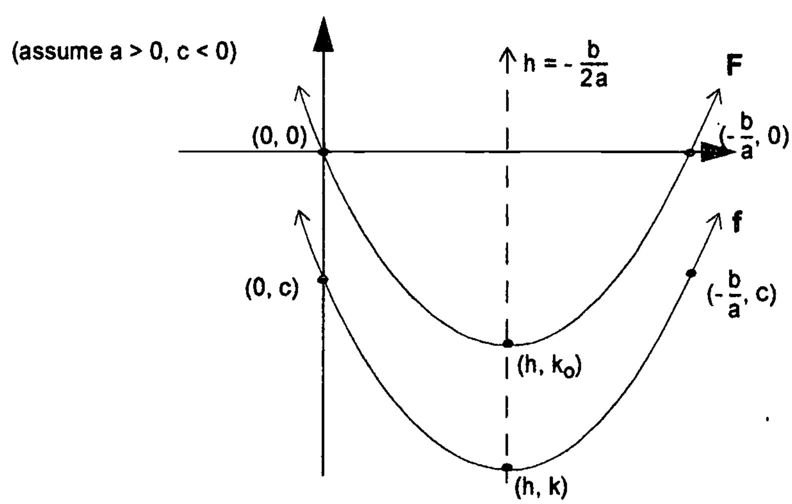
③ The axis of symmetry is $x = -\frac{b}{2a}$

④ The vertex is at $\left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$

⑤ The y intercept is at $(0, c)$ with $(-\frac{b}{a}, c)$ as a point of symmetry

⑥ The x intercepts are $\left(\frac{-b-\sqrt{b^2-4ac}}{2a}, 0\right)$ and $\left(\frac{-b+\sqrt{b^2-4ac}}{2a}, 0\right)$

Let us consider $F(x) = ax^2 + bx$ as the **base function** of $f(x) = ax^2 + bx + c$, or the original function less the constant term. Four things are immediately apparent: ① **F** and **f** share the same axis of symmetry, ② the zeroes of **F** are easily found, ③ the axis of symmetry can be determined mentally, and ④ the vertices of **F** and **f** differ only by the constant c. Consider these vertices to be (h, k_0) and (h, k) respectively, with $k = k_0 + c$.



Claim: The vertex of **F** can alternatively be given as $(h, k_o) \rightarrow (h, -ah^2)$.

Proof: Note that for **F**: ① The x intercepts are (0, 0) and $(-\frac{b}{a}, 0)$

② the axis of symmetry is halfway between these points, $h = -\frac{b}{2a}$.

Since the vertex must lie on the axis of symmetry,

③ the x coordinate of the vertex is $h = -\frac{b}{2a}$ and the y coordinate is $k_o = F(-\frac{b}{2a})$.

As shown below, the expression $F(-\frac{b}{2a})$ is equivalent to $-a\left(\frac{b}{2a}\right)^2$:

$$
\begin{aligned}
F(x) &= ax^2 + bx && \text{original function} \\
F(-\tfrac{b}{2a}) &= a(-\tfrac{b}{2a})^2 + b(-\tfrac{b}{2a}) && \text{substitute } x = -\tfrac{b}{2a} \\
&= \frac{b^2}{4a} - \frac{b^2}{2a} \\
&= \frac{b^2 - 2b^2}{4a} \\
&= \frac{-b^2}{4a} && \text{simplify} \\
&= \frac{-b^2}{4a} \cdot \frac{a}{a} && \text{multiply by } \tfrac{a}{a} \\
&= -a\left(\frac{b}{2a}\right)^2 && \text{result}
\end{aligned}
$$

From $h = -\frac{b}{2a}$ we have $-h = \frac{b}{2a}$ and it follows that

$$
\begin{aligned}
F(-\tfrac{b}{2a}) &= -a(-h)^2 && \text{substitute } -h = \tfrac{b}{2a} \\
&= -ah^2 && (-h)^2 = h^2
\end{aligned}
$$

This verifies $k_o = -ah^2$. The vertex of **F** is $(h, k_o) \rightarrow (h, -ah^2)$ as claimed. ◆

It is very significant to note that the vertex of both F(x) and f(x) *can now be determined using elementary operations on the single value h,* since $k_o = -ah^2$ and $k = k_o + c$. In other words, the vertex of the original function can be found by shifting or migrating the base function (i.e. the original function less its constant term). This is all the more valuable in that h itself can be found ***mentally*** -- it is the point halfway between the x intercepts. In light of my original goal stated in the first paragraph (the solution of function inequalities) it seemed natural to next consider the effect this vertical migration had on the x-intercepts of the original function **f**.

Claim: The x-intercepts (zeroes) of **f** are $x = h + \sqrt{-\frac{k}{a}}$ and $x = h - \sqrt{-\frac{k}{a}}$.

Proof: The zeroes of **f** are the solutions of $ax^2 + bx + c = 0$. Solve by completing the square:

$$
\begin{aligned}
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \text{divide by leading coefficient}\\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} && \text{add } \left(\frac{b}{2a}\right)^2 \text{ to each side}\\
(x + \frac{b}{2a})^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} && \text{left side factors as a binomial square}\\
(x - h)^2 &= (-h)^2 - \frac{c}{a} && \text{substitute } \frac{b}{2a} = -h\\
(x - h)^2 &= h^2 - \frac{c}{a} && (-h)^2 = h^2\\
(x - h)^2 &= -\frac{k_o}{a} - \frac{c}{a} && \text{substitute } -\frac{k_o}{a} = h^2 \text{ (from } k_o = -ah^2)\\
(x - h)^2 &= \frac{-(k_o + c)}{a} && \text{combine terms}\\
(x - h)^2 &= -\frac{k}{a} && \text{substitute } k_o + c = k\\
x - h &= \pm\sqrt{-\frac{k}{a}} && \text{solve for x}\\
x &= h \pm \sqrt{-\frac{k}{a}} && \textbf{\textit{Richard's Formula}} \text{ (see final notes)}
\end{aligned}
$$

This shows the zeroes of **f** are $x = h + \sqrt{-\frac{k}{a}}$ and $x = h - \sqrt{-\frac{k}{a}}$ as claimed. ◆

A startling result ...

A complete graph of $f(x) = ax^2 + bx + c$ can now be drawn by (mentally) migrating its base function and locating the zeroes as shown. The efficiency of this graphing alternative (***The Migration Method***) and the significance of its simplicity cannot be overstated -- as it circumvents the need to manipulate the function and increases an understanding of both symmetry and shifts. Using the method in conjunction with ***Richard's Formula*** bypasses the (student's) difficulties in using the quadratic formula, making all aspects of the quadratic function more accessible with no compromise in the integrity of the curriculum and no damage to the science or art of mathematics. In addition, it opens up a wonderful opportunity to discuss exact forms versus approximate forms.

The main ideas are summarized here, followed by several examples. For the quadratic function $f(x) = ax^2 + bx + c$ and the "base function" $F(x) = ax^2 + bx$, the axis of symmetry is $-\frac{b}{2a} = h$ and ...

	x-intercepts	vertex	y-intercept	point of symmetry
For **F**:	$(0, 0)$ and $(-\frac{b}{a}, 0)$	$(h, -ah^2)$ (h, k_o)	$(0, 0)$	$(-\frac{b}{a}, 0)$
For **f**:	(see below)	$(h, k_o + c)$ add "c" $(h, k_o + c) \rightarrow (h, k)$	$(0, c)$ add "c"	$(-\frac{b}{a}, c)$ add "c"

The x intercepts of **f** (found after locating the vertex) are given by $(h \pm \sqrt{-\frac{k}{a}}, 0)$.

Note that the y intercept and one x-intercept of **F** coincide at (0, 0). The other x-intercept can be found by inspection (since the first is at the origin). These x-intercepts can also be viewed as *the y-intercept and a point which is symmetric to the y-intercept*, reinforcing the concept of symmetry and making this point of symmetry easy (almost trivial) to find for both **F** and **f**. We begin with an example where $a = 1$ and b is an even integer. This simplifies our work even further, yielding $k_o = -h^2$ with the x-intercepts of **f** at $x = h \pm \sqrt{-k}$. *The actual graphs are left for the reader to construct, only the vital information is presented here.*

Example 1: Graph the function $f(x) = x^2 - 10x + 17$ and locate its zeroes (if they exist).

solution: For $F(x) = x^2 - 10x$, the zeroes are (0, 0) and (10, 0) by inspection, with **h = 5** (halfway point) as the axis of symmetry. Note **a = 1** and **c = 17**. It follows that:

	x-intercepts	vertex	y-intercept	point of symmetry
For **F**:	(0, 0) and (10, 0)	(5, -25) $(h, -h^2)$	(0, 0)	(10, 0)
For **f**:	(see below)	(5, -8) (h, k) $k = k_o + 17$	(0, 17) add 17	(10, 17) add 17

Since a = 1 and k = -8, the x intercepts of **f** are given by $(5 \pm \sqrt{8}, 0)$.
$(h \pm \sqrt{-k}, 0)$

If b is an odd number, there is no significant increase in difficulty when computing the "halfway point", since for any integer **z** we know that $(\mathbf{z}.5)^2 = |\mathbf{z}|(|\mathbf{z}| + 1).25$. For instance, $(\mathbf{6}.5)^2 = \mathbf{6}(\mathbf{6} + \mathbf{1}).25$ or 42.25, and $(-\mathbf{3}.5)^2 = \mathbf{3}(\mathbf{3} + 1).25$ or 12.25.

Example 2: Graph the function $f(x) = x^2 + 13x - 15$ and locate its zeroes (if they exist).

solution: For $F(x) = x^2 + 13x$, the zeroes are (0, 0) and (-13, 0) by inspection, with **h = -6.5** as the axis of symmetry. Note **a = 1** and **c = -15**. It follows that:

	x-intercepts	vertex	y-intercept	point of symmetry
For **F**:	(0, 0) and (-13, 0)	(- 6.5, - 42.25) $(h, -h^2)$	(0, 0)	(-13, 0)
For **f**:	(see below)	(- 6.5, - 57.25) (h, k) $k = k_o + (-15)$	(0, -15) add -15	(-13, -15) add -15

Since a = 1 and k = -57.25, the x intercepts of **f** are given by $(-6.5 \pm \sqrt{57.25}, 0)$.
$(h \pm \sqrt{-k}, 0)$

Even when $a \neq 1$, the *Migration Method* is more efficient than methods currently employed since virtually every step can still be done mentally -- even when students or instructors choose to work with rational values.

Example 3: Graph the function $f(x) = 2x^2 + 7x - 10$ and locate its zeroes (if they exist).

solution: For $F(x) = 2x^2 + 7x$, the zeroes are (0, 0) and $(-\frac{7}{2}, 0)$, with $\mathbf{h = -\frac{7}{4}}$ as the axis of symmetry. Note $\mathbf{a = 2}$ and $\mathbf{c = -10}$. It follows that:

	x-intercepts	vertex	y-intercept	point of symmetry
For **F**:	(0, 0) and $(-\frac{7}{2}, 0)$	$(-\frac{7}{4}, -\frac{49}{8})$ $(h, -ah^2)$	(0, 0)	$(-\frac{7}{2}, 0)$
For **f**:	(see below)	$(-\frac{7}{4}, -\frac{129}{8})$ (h, k) $k = k_o + (-10)$	(0, -10) add -10	$(-\frac{7}{2}, -10)$ add -10

Since $a = 2$ and $k = -\frac{129}{8}$, the x intercepts of **f** are given by $(-\frac{7}{4} \pm \sqrt{\frac{129}{16}}, 0)$.

$(h \pm \sqrt{-\frac{k}{a}}, 0)$

The final result can be simplified to $\frac{-7 \pm \sqrt{129}}{4}$.

One of the more remarkable benefits of determining the zeroes using *Richard's Formula*, comes when the roots of the function are complex, as in Example 4.

Example 4: Graph the function $f(x) = -2x^2 + 9x - 12$ and locate its zeroes (if they exist).

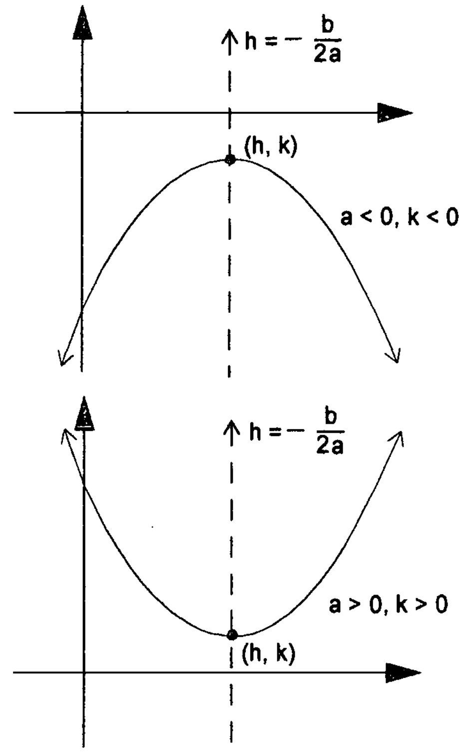
solution: For $F(x) = -2x^2 + 9x$, the zeroes are (0, 0) and $(\frac{9}{2}, 0)$ by inspection, with $\mathbf{h = \frac{9}{4}}$ as the axis of symmetry. Note $\mathbf{a = -2}$ and $\mathbf{c = -12}$. It follows that:

	x-intercepts	vertex	y-intercept	point of symmetry
For **F**:	(0, 0) and $(\frac{9}{2}, 0)$	$(\frac{9}{4}, \frac{81}{8})$ $(h, -ah^2)$	(0,0)	$(\frac{9}{2}, 0)$
For **f**:	(see below)	$(\frac{9}{4}, -\frac{15}{8})$ (h, k) $k = k_o + (-12)$	$(0, -12)$ add -12	$(\frac{9}{2}, -12)$ add -12

The complex roots of **f** are $x = \frac{9}{4} \pm \sqrt{-\frac{15}{16}}$ $\quad (x = h \pm \sqrt{-\frac{k}{a}})$,

which simplifies to $x = \frac{9}{4} \pm \frac{\sqrt{15}}{4} \boldsymbol{i}$.

Since a and k have like signs, the radicand of $\sqrt{-\frac{k}{a}}$ is negative and there are no real roots. Talk about connections! The radicand is explicitly telling a student that if the graph is concave up with k positive (a > 0 and k > 0), or if the graph is concave down with k negative (a < 0 and k < 0), the result is an imaginary number and *there are no x intercepts*. This conceivably antiquates the "old" discriminant $b^2 - 4ac$ while explicitly stating a link between characteristics of the graph and the nature of its zeroes.



In summary, I believe the Migration Method opens an exciting new chapter in the study of quadratic functions – one of the major foundations of undergraduate mathematics. The method promotes a higher degree of conceptualization, reinforces the concept of symmetry, encourages mental visualization and strengthens the ability to graph a given function by translating a parent or base function. Additionally, I believe the method will lead to a quicker solution and a better understanding of quadratic and other inequalities.

Final Note

I've named this new formula for solving a quadratic ***Richard's Formula***, in honor of my father. As an instructor of mathematics, there were few who were his equal.

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Signature: John W. Coburn	Printed Name/Position/Title: John W Coburn Professor of Math
Organization/Address:	Telephone: 314-595-4343 FAX: 314-595-2199
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